DRAFT VERSION FEBRUARY 2, 2008
Preprint typeset using LATEX style emulateapj v. 04/03/99

HYDROMAGNETIC AND GRAVITOMAGNETIC CRUST-CORE COUPLING IN A PRECESSING NEUTRON STAR

YURI LEVIN[1] AND CAROLINE D'ANGELO[1]
Canadian Institute for Theoretical Astrophysics, 60 St. George Street, Toronto, ON M5S 3H8, Canada

Draft version February 2, 2008

ABSTRACT

We consider two types of mechanical coupling between the crust and the core of a precessing neutron star. First, we find that a hydromagnetic (MHD) coupling between the crust and the core strongly modifies the star's precessional modes when $t_a \leq\sim (T_s \times T_p)^{1/2}$; here t_a is the Alfven crossing timescale, and T_s and T_p are the star's spin and precession periods, respectively. We argue that in a precessing pulsar PSR B1828-11 the restoring MHD stress prevents a free wobble of the crust relative to the non-precessing core. Instead, the crust and the proton-electron plasma in the core must precess in unison, and their combined ellipticity determines the period of precession. Link has recently shown that the neutron superfluid vortices in the core of PSR B1828-11 cannot be pinned to the plasma; he has also argued that this lack of pinning is expected if the proton Fermi liquid in the core is type-I superconductor. In this case, the neutron superfluid is dynamically decoupled from the precessing motion. The pulsar's precession decays due to the mutual friction between the neutron superfluid and the plasma in the core. The decay is expected to occur over tens to hundreds of precession periods and may be measurable over a human lifetime. Such a measurement would provide information about the strong n-p interaction in the neutron-star core.

Second, we consider the effect of gravitomagnetic coupling between the neutron superfluid in the core and the rest of the star and show that this coupling changes the rate of precession by about 10%. The general formalism developed in this paper may be useful for other applications.

Subject headings: neutron stars

1. INTRODUCTION

The most conclusive evidence for a free precession of an isolated pulsar comes from Stairs, Lyne, and Shemar (2000, SLS); see also Cordes 1993 and Shabanova, Lyne, and Urama 2001. Their discovery has shown convincingly that some pulsars are precessing, and has opened a new window into the interior of neutron stars (Link and Epstein 2001, Jones and Andersson 2001, Link and Cutler 2002, Cutler, Ushomirsky and Link 2003, Wasserman 2003, Link 2003; see also Link 2002 for a review). The pulsar PSR B1828-11, which has been monitored by SLS for about a decade, is spinning with the period of 0.4 seconds and precessing with the period of 500 or 1000 days. The large ratio of the precession to spin periods is difficult to reconcile with the current theoretical ideas about the neutron star's internal structure. In particular, it has long been argued that the neutron superfluid vortices are pinned to the crystal lattice of the crust; this has been used to explain pulsar glitches (sudden spin-ups of young isolated pulsars). However, as was shown in the pioneering work of Shaham (1977), the crustal pinning leads to rather short precession periods, $T_{\rm precession} = (I_{\rm star}/I_{\rm superfluid})T_{\rm spin}$. Here $T_{\rm spin}$ and $T_{\rm precession}$ are the spin and the precession periods respectively, $I_{\rm star}$ and $I_{\rm superfluid}$ are the moments of inertia of the star and the pinned superfluid respectively. The expected precession period of PSR B1828-11 would be of the order of 100 seconds, in sharp contrast with what has been observed. Link and Cutler (2001) have proposed a way out of this contradiction: they argue that the observed precession is so strong that the superfluid vortices are unpinned from and move freely through the crustal lattice. Another possibility is that the vortices in the superfluid in the crust are never pinned to the crustal lattice; this has been argued on theoretical grounds by Jones (1998). We feel that both of these ideas, while potentially viable, require more detailed calculations.

In addition, if the proton-electron plasma in the core participate in the precessing motion and if, as is commonly believed, the protons condense into type-II superconductor (Baym, Pethick, and Pines 1969), then the expected strong interaction between the superconductor's fluxtubes and the vortices of the neutron superfluid does not allow slow precession with small damping (Link 2003). The conflict with observations is avoided if either : 1. the proton-electron plasma does not participate in the precessing motion, and the crust alone precesses ("Chandler wobble"); we will show that this possibility is excluded due to the MHD crust-core coupling, or 2. the protons in the core do not form type-II superconductor, as is commonly believed, but instead they form a type-I superconductor (Link 2003). This is not far-fetched since proton pairing calculations in the core are uncertain; moreover, recent work by Buckley, Petlitski and Zhitnitsky (2003) argues that the interaction between the proton and neutron condensates may turn the neutron-star interior into type-I superconductor even if the proton pairing calculations favor the type-II phase.

While it would be exciting to gain some observational handle on the exotic quantum fluids in the neutron-star interior, the main focus of this paper is elsewhere. Here, we concentrate instead on the MHD and gravitomagnetic coupling between the crust and the core; these effects make an impact on the precession dynamics yet their nature is well-understood theoretically and (in case of MHD) is well-tested in laboratory experiments. We will, however, also discuss the decay of pulsar precession due to the mutual friction between the neutron superfluid and the proton-electron plasma in the core.

This paper is structured as follows. In section II we present a toy model for MHD coupling between the crust and the core and solve the Euler's precession equations within this model.

We show that the precession period is strongly affected once the timescale of magnetic coupling is comparable to the geometric mean between the spin and precession periods. In this section we also consider the damping of precession via electron scattering off the magnetized neutron superfluid vortices in the core (this precess is called mutual friction). The damping timescale is generally tens or hundreds of precession periods, and its exact value is sensitive to the effective proton mass in the core. Thus by monitoring the precession decay over a few decades one may be able to constrain the strong n-p interactions in the neutron-star core.

In Section III we move away from toy model and consider the nature of slow MHD waves in a rotating gravitationally stratified neutron-star core. Our calculations for this more realistic model generally confirm our toy-model results.

Finally, in section IV we take into account relativistic frame-dragging around spinning neutron star, and find two modes of relativistic precession. The first mode is the Lense-Thirring (LT) precession of the crust in the gravito-magnetic field of the core, first considered by Blandford and Coppi (Blandford 1995). The Lense-Thirring precession is relatively fast; its period is only an order of magnitude greater than the spin period of the pulsar. The LT precession is most easily excited when the crust suddenly changes its angular momentum (e.g. due to a collision with an asteroid). It is damped on the timescale of viscous coupling between the crust and the core, about 100 seconds.

The second mode is the Eulerian precession which is modified by the inertial frame dragging. This mode is most easily excited by a change in the crustal tensor of inertia, e.g. by a sudden deformation of the crust due to magnetic forces. We find that the frame dragging modifies the precession frequency by about 10%.

2. IDEALIZED MODEL FOR THE CRUST-CORE MHD COUPLING.

Ohmic dissipation inside the neutron star is very slow compared to the precession period, and therefore ideal MHD provides an excellent description of the neutron-star interior. The magnetic field threads both the crust and the core; in an ideal MHD the relative displacement of the crust and the core creates magnetic stresses which oppose this displacement. Thus, the magnetic field lines act as elastic strings (e.g., Blandford and Thorne 2004). Motivated by this we follow the spirit of Bondi and Gold's (1955, BG) analysis of the Chandler Wobble, and consider the crust and the core as solid bodies coupled by a torque which opposes their relative displacement:

$$\vec{\tau} = -\mu\delta\vec{\phi}. \tag{1}$$

Here $\delta\vec{\phi}$ is the small angular displacement between the crust and the core, and μ is a constant representing the strength of MHD coupling. While this model is simplistic, it is (a) fully solvable and (b) correct in predicting the main features of the precession with MHD crust-core coupling. We consider a more realistic model in the next section.

For mathematical simplicity, we assume the crust is axisymmetric and we work in the coordinate system $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ rigidly attached to the crust so that $\vec{e}_3$ is directed along the symmetry axis. We also assume the core to be spherically symmetric. We denote by (A, A, C) and (D, D, D) the crust's and the core's three principal moments of inertia, respectively. The dynamics of the system is described by the coupled Euler's equations [cf. Eq. (1) of BG]:

$$\begin{aligned} A\dot{\omega}_1 + (C-A)\omega_2\omega_3 &= -\mu\delta\phi_1 = -D(\dot{\Omega}_1 + \omega_2\Omega_3 - \omega_3\Omega_2), \\ A\dot{\omega}_2 - (C-A)\omega_1\omega_3 &= -\mu\delta\phi_2 = -D(\dot{\Omega}_2 - \omega_1\Omega_3 + \omega_3\Omega_1), \\ C\dot{\omega}_3 &= -\mu\delta\phi_3 = -D\dot{\Omega}_3. \end{aligned} \tag{2}$$

Here $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\vec{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ are the angular velocity vectors of the crust and the core, respectively, and the sign of $\delta\vec{\phi}$ is chosen so that

$$\frac{d\delta\vec{\phi}}{dt} = \vec{\omega} - \vec{\Omega}. \tag{3}$$

The observed wobble angle of PSR B1828-11 is only ~ 3 degrees (Link and Epstein 2001); motivated by this we restrict our analysis to small-amplitude precession. More precisely, we consider a small periodic perturbation of an equilibrium state in which both the crust and the core are rotating around the crust's symmetry axis with the angular velocity n. The dynamical quantities are then expressed as follows:

$$\begin{aligned} \omega_1 &= \tilde{\omega}_1 e^{i\sigma t}, \\ \omega_2 &= \tilde{\omega}_2 e^{i\sigma t}, \\ \omega_3 &= n + \tilde{\omega}_3 e^{i\sigma t}, \end{aligned} \tag{4}$$

and analogously,

$$\begin{aligned} \Omega_1 &= \tilde{\Omega}_1 e^{i\sigma t}, \\ \Omega_2 &= \tilde{\Omega}_2 e^{i\sigma t}, \\ \Omega_3 &= n + \tilde{\Omega}_3 e^{i\sigma t}. \end{aligned} \tag{5}$$

Here it is assumed that the complex amplitudes $\tilde{\omega}_i$ and $\tilde{\Omega}_i$ are small compared to n. It is also convenient to define, in the usual way, the crust's ellipticity:

$$\epsilon = \frac{C-A}{A}. \tag{6}$$

In the dynamical Equations (2) we can neglect the terms which are of second order with respect to $\tilde{\omega}_i$ and $\tilde{\Omega}_i$, and use Eq. (3) to eliminate $\delta\vec{\phi}$. The linearized equations of motion are given below:

$$\begin{aligned} i\sigma A\tilde{\omega}_1 + \epsilon An\tilde{\omega}_2 &= \frac{\mu}{i\sigma}(\tilde{\Omega}_1 - \tilde{\omega}_1) = -iD\sigma\tilde{\Omega}_1 - Dn(\tilde{\omega}_2 - \tilde{\Omega}_2), \\ i\sigma A\tilde{\omega}_2 - \epsilon An\tilde{\omega}_1 &= \frac{\mu}{i\sigma}(\tilde{\Omega}_2 - \tilde{\omega}_2) = -iD\sigma\tilde{\Omega}_2 + Dn(\tilde{\omega}_1 - \tilde{\Omega}_1), \\ i\sigma C\tilde{\omega}_3 &= \frac{\mu}{i\sigma}(\tilde{\Omega}_3 - \tilde{\omega}_3) = -iD\sigma\tilde{\Omega}_3. \end{aligned} \tag{7}$$

The third equation above is decoupled from the first two; it describes the small rotations of the crust and the core around the symmetry axis of the crust. This equation alone gives two frequency eigenvalues:

$$\begin{aligned} \sigma_3 &= 0, \\ \sigma_4 &= \left[\frac{\mu(C+D)}{CD}\right]^{1/2}. \end{aligned} \tag{8}$$

The trivial eigenvalue σ_3 corresponds to the crust and the core rotating in unison without any relative displacement, whereas the eigenvalue σ_4 corresponds to the crust-core oscillations around the rotation axis; these oscillations are not affected by

the ellipticity of the crust and the rate of stellar rotation, and do not represent precession. The information about precession is contained in the first two equations of (7). Following BG, we can simplify the algebra by considering the sum (first equation)+$i\times$(second equation), and by introducing the new variables, $\omega^+ = \tilde{\omega}_1 + i\tilde{\omega}_2$ and $\Omega^+ = \tilde{\Omega}_1 + i\tilde{\Omega}_2$. In the end, we get the following eigenvalue equation:

$$(\sigma - \epsilon n)(\sigma^2 + n\sigma) = \mu\left(\frac{A+D}{AD}\sigma - \epsilon\frac{n}{D}\right). \tag{9}$$

This equation has three solutions:

$$\sigma_0 \simeq -n - \sigma_m^2/n, \tag{10}$$

$$\sigma_{1,2} \simeq \left[\sigma_p \pm \sqrt{\sigma_p^2 - 4\sigma_d^2}\right]/2, \tag{11}$$

where

$$\sigma_m = \sqrt{\frac{\mu(A+D)}{AD}}, \tag{12}$$

is the frequency of the mode in which the crust and the core of a non-rotating star oscillate differentially, with the restoring force of purely MHD origin, and

$$\sigma_p = \epsilon n + \sigma_m^2/n, \tag{13}$$

$$\sigma_d = \sqrt{\frac{\mu\epsilon}{D}}.$$

Since in our case $A \ll D$, one can show that $\sigma_p \gg 2\sigma_D$ for all values of ϵ and μ. We therefore have

$$\sigma_1 \simeq \sigma_p = \epsilon n + \sigma_m^2/n, \tag{14}$$

$$\sigma_2 \simeq \frac{A}{A+D}\epsilon n(1 + \epsilon n^2/\sigma_m^2)^{-1}. \tag{15}$$

The frequency σ_1 characterizes the differential precession between the crust and the core; in the limit of zero magnetic coupling (i.e., $\sigma_m = 0$) its value $\sigma_1 = n\epsilon$ is the frequency of a free precession of the crust. By contrast, the frequency σ_2 corresponds to the mode in which the crust and the core are trying to precess in unison. In the limit of infinite magnetic coupling (i.e., $\sigma_m = \infty$) its value of $\sigma_2 = \epsilon nA/(A+D)$ is the frequency of precession of the neutron star as a whole: the crust is the source of ellipticity but the core is rigidly attached to the crust and they precess together.

Let's apply these results to PSR B1828-11. The inferred dipole magnetic field of this pulsar is $B \simeq 5\times 10^{12}$G (see SLS), and the Alfven speed in the core is

$$v_a = 10^5\left(\frac{B}{5\times 10^{12}\mathrm{G}}\right)\left(\frac{2\times 10^{14}\mathrm{g/cm^3}}{\rho}\right)^{1/2} \mathrm{cm/s} \tag{16}$$

when the core is not superconducting, and

$$v_a = 1.4\times 10^6\left(\frac{B}{5\times 10^{12}\mathrm{G}}\right)^{1/2}\left(\frac{2\times 10^{14}\mathrm{g/cm^3}}{\rho}\right)^{1/2} \mathrm{cm/s} \tag{17}$$

when the core is superconducting. Here ρ is the density of the core material which is participating in the Alfven-wave motion. We estimate the characteristic $\sigma_m \sim \pi v_a/R$ to be $0.3\mathrm{s}^{-1}$ for a non-superconducting core, and $4.2\mathrm{s}^{-1}$ for a superconducting core. In deriving these numbers we have assumed that all of the core is participating in the Alfven-wave motion; we remark that if the neutrons form a superfluid, then only the charged proton-electron plasma is magnetically coupled to the crust, and the estimates for σ_m should increase by a factor of ~ 4. The spin period of PSR B1828-11 is $T_{\rm spin} \simeq 0.4$s, and from Eq. (14) we see that the period of the crust-core differential precession (Chandler Wobble) is

$$T_1 = 2\pi/\sigma_1 \sim 10^3\mathrm{s} \tag{18}$$

for a non-superconducting core, and

$$T_1 \sim 5\mathrm{s} \tag{19}$$

for a superconducting core. In the above estimates, we have assumed zero ellipticity for the star and that all of the core is magnetically coupled to the crust; thus our estimates are upper limits on T_1. Since the precession period of PSR B1828-11 is $\sim 4\times 10^7$s, we can say with certainty that the observed precession is not the "Chandler wobble" of the crust relative to the core. Rather, in agreement with the argument sketched by Link (2003), the crust and the magnetically-coupled part of the core precess in unison[1] and their precession period is found from Eq. (15):

$$T_2 \simeq \frac{T_{\rm spin}}{\epsilon}\frac{A+D}{A} = 8\times 10^7\left(\frac{T_{\rm spin}}{0.4\mathrm{s}}\right)\left(\frac{10^{-7}}{\epsilon}\right)\left(\frac{1+D/A}{20}\right)\mathrm{s}. \tag{20}$$

In deriving Equations (15) and (20), we have assumed that there are no extra torques acting on the charged plasma of the core. This assumption breaks down if the core is a type-II superconductor and the neutron superfluid vortices interact strongly with the magnetic fluxtubes. Link (2003) has shown that a strong vortex–fluxtube interaction is inconsistent with the observed precession on PSR B1828-11, but has pointed out that the difficulty is alleviated if the core superconductivity is of type-I rather than type-II. In this case, the magnetic field is contained not in quantized fluxtubes but in larger domains (although these domains probably still thread densely the neutron-star interior). Then, the relative motion of the plasma and the neutron superfluid is damped by the scattering of the electrons on the magnetized supefluid vortices [Alpar, Langer, and Sauls 1984, Alpar and Sauls 1988]. This damping is known as "mutual friction", and its characteristic timescale is

$$t_{\rm mf} = 10T_{\rm spin}(m_p/\delta m_p^*)^2 f(m_p^*, m_n^*, \Delta_n, \rho_c, \rho), \tag{21}$$

cf. Eq. (32) of Alpar, Langer, and Sauls. Here m_p/m_n and m_p^*/m_n^* are the bare and the effective proton/neutron masses, Δ_n is the neutron condensate gap, ρ_c is the density of the proton-electron plasma in the core, and f is a function which depends weakly on its variables. Alpar and Sauls (1988) give detailed discussion of $t_{\rm mf}$, and we refer to them for the details.

[1] Link (2003) has argued that the crust and the charged part of the core precess in unison when the precession frequency is $\omega_{\rm prec} < \sigma_m$. However, this estimate does not take into account the rotation of the star; we see from our Eq. (14) that the correct criterion is $\omega_{\rm prec} < \sigma_m^2/n$, a more stringent condition.

The precession damping timescale $\tau_{\rm pr}$ is determined by the following relation[2] (see, e.g., BG):

$$\tau_{\rm pr} = T_2 t_{\rm mf}/T_{\rm spin} \sim 15(m_p/\delta m_p^*)^2 \text{years}. \qquad (22)$$

Thus, the mutual-friction damping of precession may be observable for PSR B1828-11 over the timescale of human life, and its measurement will yield the information about $\delta m_p^*/m_p$.

3. ALFVEN WAVES IN A ROTATING NEUTRON STAR.

It is interesting to note that in the absence of the crust ellipticity, the frequency of the neutron-star Chandler Wobble scales as $1/n$, see Eq. (14). As is seen from this equation the fast rotation reduces the effectiveness of MHD crust-core coupling. For a neutron star with a fluid core the coupling is mediated by the Alfven waves which are excited by the precessing crust and propagate into the core. The characteristic timescale for the coupling is $\sim R/v_{ap}$, where v_{ap} is the speed of these Alfven waves. Thus, we expect that the Alfven waves are slowed down as the star spins faster; our detailed analysis below confirms this expectation.

MHD in rotating fluids has been the subject of an extensive research in geophysical fluid dynamics, with applications to the Earth's fluid core [see Hide, Boggs, and Dickey (2000) and references therein.] There is, however, a significant difference between the Earth and neutron star cores. The Earth interior is approximately isenthropic, and the Taylor-Proudman theorem is applicable; thus the velocity field is almost constant along the lines parallel to the rotation axis. By contrast, the neutron-star interior is stable stratified due to the core's radial composition gradient (Reisenegger and Goldreich, 1992). The fluid motion is restricted to equipotential shells, which we assumed to be spherical (this is a good approximation for the slowly-spinning PSR B1828-11). The motion is strongly subsonic, therefore the velocity field is divergence-free.

Under these conditions, the general small fluid displacements can be represented by the radius-dependent stream function $\psi(r,\theta,\phi)$, so that in spherical coordinates the displacement components are

$$\begin{aligned} \zeta_r &= 0, \\ \zeta_\phi &= \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \\ \zeta_\theta &= -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \end{aligned} \qquad (23)$$

The radial component of the vorticity of the fluid is

$$\eta = (1/r)^2(\partial/\partial t)\nabla_r^2\psi, \qquad (24)$$

where ∇_r^2 is the Laplacian operator on the unit sphere:

$$\nabla_r^2 = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right]. \qquad (25)$$

Since the fluid inside the neutron star is strongly stratified by gravity (i.e., the Brunt-Vaissalla frequency $N \gg \Omega, \sigma_m$), the fluid motion on different shells is coupled only via magnetic stresses. One can write down the dynamical equation for the radial component of the absolute vorticity, cf. Eq. (5) of Levin and Ushomirsky (2001):

$$\frac{d}{dt}(\eta + 2\Omega\cos\theta) = \hat{r}\cdot\nabla\times\vec{a}_B. \qquad (26)$$

Here $d/dt = \partial/\partial t + \vec{v}\cdot\nabla$ is the Lagrangian time derivative, $\hat{r}$ is the unit radial vector and $\vec{a}_B$ is the acceleration due to the restoring magnetic stress. Following Kinney and Mendell (2002), we restrict ourselves to the special case of the spherically symmetric radial magnetic field, $\vec{B} = B(r)\hat{r}$. The results obtained below should be qualitatively correct for a more general field configuration; however, we have chosen a particularly simple geometry in which the mathematical evaluation of the right-hand side in Eq. (26) is greatly simplified. The relevant components of $\vec{a}_B$ are given by

$$\vec{a}_B\cdot\vec{e}_{\theta,\phi} = \frac{1}{4\pi\rho r}\frac{\partial}{\partial r}\left[B^2r^2\frac{\partial}{\partial r}\left(\frac{\zeta_{\theta,\phi}}{r}\right)\right]. \qquad (27)$$

We can now write down the linearized equation of motion for the stream function:

$$\frac{\partial^2}{\partial t^2}\nabla_r^2\psi + 2\Omega\frac{\partial}{\partial t}\frac{\partial}{\partial\phi}\psi = \frac{1}{4\pi\rho}\frac{\partial}{\partial r}\left[B^2r^2\frac{\partial}{\partial r}\left(\frac{\nabla^2\psi}{r^2}\right)\right]. \qquad (28)$$

We look for the solution of Eq. (28) in the following form:

$$\psi(r,\theta,\phi) = \Sigma_{l,m}\psi_{lm}(r)Y_{lm}(\theta,\phi)e^{i\sigma_{lm}t}. \qquad (29)$$

Since Y_{lm} is an eigenfunction of both $\partial/\partial\phi$ and ∇_r^2, Eq. (28) separates into individual ordinary differential equations for ψ_{lm}:

$$\left[\sigma_{lm}^2 - \frac{2m\Omega\sigma_{lm}}{l(l+1)}\right]\psi_{lm}(r) + \frac{1}{4\pi\rho}\frac{\partial}{\partial r}\left[B^2r^2\frac{\partial}{\partial r}\left(\frac{\psi_{lm}(r)}{r^2}\right)\right] = 0. \qquad (30)$$

We now consider the short-wavelength (WKB) approximation for the above equation, and hence derive the following dispersion relation:

$$k^2 = \frac{1}{v_a^2}\left[\sigma_{lm}^2 - \frac{2m\Omega\sigma_{lm}}{l(l+1)}\right], \qquad (31)$$

where k is the radial wavevector. The purely toroidal Alfven waves correspond to the case when $m = 0$ in the above equation. These waves are not affected by the stellar rotation and have the dispersion relation identical to that of Alfven waves in a non-rotating star. However, in PSR B1828-11 the Alfven waves are excited by the slowly precessing crust, and therefore one should consider the waves with $l = 1$ and $m = -1$. In this case the second term on the right-hand side of Eq. (31) is the dominant one since $\sigma \ll \Omega$, and the wave is strongly slowed down by the stellar rotation, just as we expected. The radial wavelength of the excited Alfven mode is

$$\lambda_a = 2\pi/k \simeq 2\pi v_a/\sqrt{\sigma\Omega}. \qquad (32)$$

which equals $\sim 6\times10^8$cm and $\sim 8\times10^9$cm for normal and for superconducting neutron star interior, respectively. In both cases, it is more than two orders of magnitude greater than the radius of neutron star, $\sim 10^6$cm. Therefore, the part of the neutron-star interior which is magnetically coupled to the solid crust will precess in unison with the crust. This conclusion is robust and is in agreement with our toy-model results from the previous section.

[2] Alpar and Sauls (1988) have erroneously overestimated the precession damping timescale by a factor ρ/ρ_c. They have associated the viscous damping timescale with $t_{\rm mf}\rho/\rho_c$, since this is the timescale it takes for the neutron superfluid to come to co-rotation with the charged plasma. However, the neutron superfluid carries most of the star's moment of inertia, and if the superfluid spins at a different rate than the rest of the star, it is the crust+plasma which are coming to co-rotation with it. Therefore one should use $t_{\rm mf}$ for the viscous damping timescale when one evaluates the precession damping timescale.

4. GRAVITOMAGNETIC COUPLING BETWEEN THE CRUST AND THE CORE.

The gravitational redshift at a neutron-star surface is ~ 0.3, and therefore relativistic effects, including the dragging of the inertial frames, are strong in and around neutron stars. In this section we analyze how the frame-dragging affects the relative precession of the crust and the core. Our post-Newtonian calculations rely on the usage of the gravitomagnetic field, $\vec{H}$, see Thorne, Price, and MacDonald (1986) for the details of this formalism.

4.1. *The gravitomagnetic coupling torque*

Consider the gravitomagnetic force acting on the small region of the crust of mass dm[3]. In the post-Newtonian approximation, it is given by

$$d\vec{F}_{\rm GM} = dm\vec{v}\times\vec{H} = dm(\vec{\omega}\times\vec{r})\times\vec{H}, \tag{33}$$

where $\vec{v}$, $\vec{H}$, $\vec{\omega}$, $\vec{r}$ are the velocity of the small region, the gravitomagnetic field, the instantaneous angular velocity of the crust, and the radius-vector of the region, respectively. The torque acting on this region of the crust is given by

$$d\vec{T} = \vec{r}\times d\vec{F}_{\rm GM} = dm(\vec{H}\cdot\vec{r})\vec{\omega}\times\vec{r}, \tag{34}$$

where we have used the vector identity $(A\times B)\times C = (C\cdot A)B-(C\cdot B)A$. The field $\vec{H}$ is that of a dipole, and

$$\vec{H}\cdot\vec{r} = -(4/r^3)\vec{J}\cdot\vec{r} = -(4D/r^3)\vec{\Omega}\cdot\vec{r}, \tag{35}$$

where $\vec{J}$, $\vec{\Omega}$, and D are the angular momentum, the angular velocity, and the moment of inertia of the spherical core [Here we ignore interaction of the crust with its own gravitomagnetic field. It can be shown (Thorne and Gursel, 1983) that this self-interaction can be absorbed into the free precession.]

Now, substituting Eq. (35) into Eq. (34), and integrating over the crust, we arrive to the following form of the gravitomagnetic torque:

$$\vec{T}_{\rm gm} = \vec{\omega}\times I_{\rm gm}\vec{\Omega}, \tag{36}$$

where $I_{\rm gm}$ is the linear operator (represented, generally, by a 3×3 matrix) defined as follows:

$$I_{\rm gm}\vec{\Omega} = -\int d^3r\rho(\vec{r})(4D/r^3)(\vec{\Omega}\cdot\vec{r})\vec{r}. \tag{37}$$

We use Dirac's the bra and ket notation and express this operator as

$$I_{\rm gm} = -\int d^3r\rho(\vec{r})(4D/r^3)|\hat{r}><\hat{r}|. \tag{38}$$

From the above expression we see that $I_{\rm gm}$ is a hermitian operator: since the integrand $\propto |\hat{r}><\hat{r}|$ in Eq. (38) is hermitian, the integral must also be hermitian. This means that the matrix representing $I_{\rm gm}$ is symmetric.

If the crust is spherically symmetric, then $I_{\rm gm} = pI$, where p is a real number and I is a unit matrix. In this case, the torque acting on the crust is

$$\vec{T} = p\vec{\omega}\times\vec{\Omega}, \tag{39}$$

which is the familiar form of the Lense-Thirring torque acting on the gyroscope. In the situation considered here the crust is slightly deformed, so that

$$I_{\rm gm} = pI + \epsilon pK, \tag{40}$$

where K is a 3×3 matrix with entries of order 1.

[3]In this section we shall collectively refer to the solid crust and the core plasma coupled to it as the "crust".

4.2. *The dynamics of relativistic precession*

Again, we use the BG approach to the precession of an interacting crust and core. The equations of motion which include the gravitomagnetic torque components T_1, T_2, T_3, are [cf. Eq. (1) of BG]:

$$\begin{aligned}
A\dot{\omega_1} + (C-A)\omega_2\omega_3 &= \lambda(\Omega_1-\omega_1) + T_1 = -D[\dot{\Omega_1} + \omega_2\Omega_3 - \omega_3\Omega_2],\\
A\dot{\omega_2} - (C-A)\omega_3\omega_1 &= \lambda(\Omega_2-\omega_2) + T_2 = -D[\dot{\Omega_2} + \omega_3\Omega_1 - \omega_1\Omega_3],\\
C\dot{\omega_3} &= \lambda(\Omega_3-\omega_3) + T_3 = -D[\dot{\Omega_3} + \omega_1\Omega_2 - \omega_2\Omega_1].
\end{aligned} \tag{41}$$

Here we have added the terms $\lambda(\vec{\Omega}-\vec{\omega})$ which represent the viscous torque between the crust and the core (e.g., due to mutual friction between the neutron superfluid and the core plasma coupled to the solid crust). As in the previous sections, we are interested in the small amplitude precession, when the motion differs only slightly from the rigid motion rotation about z-axis, so that ω_1, ω_2, $\tilde{\omega}_3 = \omega_3 - n$, Ω_1, Ω_2, and $\tilde{\Omega}_3 = \Omega_3 - n$ are all much less than n. Then Eq. (41) becomes

$$\begin{aligned}
A\dot{\omega_1} + (C-A)n\omega_2 &= \lambda(\Omega_1-\omega_1) + T_1 = -D[\dot{\Omega_1} + n(\omega_2-\Omega_2)],\\
A\dot{\omega_2} - (C-A)n\omega_1 &= \lambda(\Omega_2-\omega_2) + T_2 = -D[\dot{\Omega_2} + n(\Omega_1-\omega_1)],\\
C\dot{\omega_3} &= \lambda(\tilde{\Omega}_3-\tilde{\omega}_3) + T_3 = -D\dot{\tilde{\Omega}}_3.
\end{aligned} \tag{42}$$

The general strategy now is to identify the leading terms in T_1, T_2, and T_3, using Eq. (36), and then solve Eq. (42). Since $D \gg A$, we consider a simplified case when the spherical core has an infinite inertia: $D\to\infty$, so that the core's spin does not change in the inertial frame of reference. Therefore we have, from Eq. (42),

$$\dot{\Omega}_1 + n(\omega_2-\Omega_2) = \dot{\Omega}_2 - n(\omega_1-\Omega_1) = \dot{\tilde{\Omega}}_3 = 0. \tag{43}$$

We look for a mode with the growth rate γ, so that $\dot{\Omega}_1 = \gamma\Omega_1$, etc. By considering the sum [the first component of Eq. (43)]+$i\times$[the second component of Eq. (43)], we get

$$\Omega^+ = \frac{in}{in+\gamma}\omega^+, \tag{44}$$

where $\Omega^+ = \Omega_1 + i\Omega_2$, $\omega^+ = \omega_1 + i\omega_2$.

Let us restrict ourselves to the case of the axially symmetric crust. In this case, both tensor of inertia and $I_{\rm gm}$ diagonalize in the same basis because of the axial symmetry. We can then write

$$I_{\rm gm}\vec{\Omega} = p\Omega_1\vec{e}_1 + p\Omega_2\vec{e}_2 + p(1+\epsilon k)\Omega_3\vec{e}_3, \tag{45}$$

where $\vec{e}_{1,2,3}$ are the unit vectors along x, y, and z axes respectively, with the z axis chosen to be the axis of symmetry [as in Eq. (42)], and k is a number of order 1. Then the gravitomagnetic torque in Eq. (36), to leading order, is

$$\vec{T} = np\left\{[\omega_2(1+\epsilon k)-\Omega_2]\vec{e}_1 + [\Omega_1-\omega_1(1+\epsilon k)]\vec{e}_2\right\}. \tag{46}$$

Now, let us substitute this expression into Eq. (42), add [first row]+$i\times$[second row], and ignore the right-hand side with D in

it [we have already taken care of it by setting $D \to \infty$]. We get, after dividing by A, and substituting γ instead of the time derivative:

$$\gamma\omega^+ + in\epsilon[1-(p/A)k]\omega^+ = (1/A)[\lambda + inp](\Omega^+ - \omega+). \quad (47)$$

Notice that in the above equations the contribution due to the gravitomagnetic terms can be represented by an effective modification of the ellipticity $\epsilon \to \epsilon[1-(p/A)k]$ and of the viscous coupling coefficient $\lambda \to \lambda + inp$. Therefore one can consider the precession solution without gravitomagnetic terms and then substitute the ellipticity and coupling in this solution by their modified values. The resulting expressions then represent the precession solution which includes the gravitomagnetic coupling.

The remaining calculation is straightforward. Let $\bar\epsilon = \epsilon[1-(p/A)k]$, $\bar\lambda = (1/A)[\lambda + inp]$. By substituting Eq. (44) into Eq. (47), we get the following equation for the growth rate γ:

$$\gamma^2 + [in(1-\bar\epsilon) + \bar\lambda]\gamma + n^2\bar\epsilon = 0, \quad (48)$$

which has two solutions:

$$\gamma = (1/2)\left\{-[in(1-\bar\epsilon)+\bar\lambda] \pm \sqrt{[in(1-\bar\epsilon)+\bar\lambda]^2 - 4n^2\bar\epsilon}\right\}. \quad (49)$$

We now use the fact that $\bar\epsilon \ll 1$ in Eq. (49), and we get to the leading order in $\bar\epsilon$ for the "+" solution

$$\gamma = i\frac{n\bar\epsilon}{1-i(\bar\lambda/n)} \simeq in\bar\epsilon - \bar\lambda\bar\epsilon. \quad (50)$$

This solution corresponds to the Eulerian precession of the crust, with the frequency

$$\omega_{\rm pr} = n\bar\epsilon - \bar\epsilon n(p/A) = n\epsilon[1-(p/A)k][1-(p/A)], \quad (51)$$

and the damping rate

$$1/\tau_{\rm pr} = (\lambda/A)\epsilon[1-(p/A)]. \quad (52)$$

The contribution of the frame dragging comes in through terms which contain p/A. The frame dragging reduces both the precession frequency and the damping rate by relative order of p/A. Now, from Eq. (39), we can work out that $p/A = \omega_{\rm LT}/n$, where$\omega_{\rm LT}$ is the frequency of the conventional, gyroscopic Lense-Thirring precession. The calculations of Blandford and Coppi (Blandford 1995) show that $\omega_{\rm LT}/n \sim 1/7$. Therefore, we expect the dragging of inertial frames reduces the frequency and the damping rate of precession by $\sim 10\%$.

What about the second, "–" solution of Eq. (48)? We have, to the leading order in $\bar\epsilon$

$$\gamma = -in - \bar\lambda = -in(1+p/A) - \lambda/A. \quad (53)$$

This mode corresponds to the situation when the spin of the crust is misaligned with that of the core. In the inertial frame of reference (as opposed to the frame attached to the body), one must take the $-in$ out of γ. The piece that is left is then

$$\gamma_{\rm inertial} = -inp/A - \lambda/A. \quad (54)$$

This corresponds to the Lense-Thirring precession considered by Blandford and Coppi (Blandford 1995), which is damped on a short timescale λ/A, i.e. the viscous time on which co-rotation of the crust and the core is enforced.

5. DISCUSSION

In this paper we have analyzed the effect of the crust-core coupling on the Chandler Wobble of the neutron-star crust. We have found (section IV) that the gravitomagnetic crust-core coupling does not affect strongly the Chandler Wobble, but instead modifies its frequency by about 10%. By contrast, we have found that the MHD interaction between the crust and the core of a rotating neutron star dramatically changes the dynamics of the Wobble, for typical values of the pulsar spin and magnetic field. In particular, we have shown that the observed precession in PSR B1828-11 can not be the Chandler Wobble of its crust; instead, the crust and the plasma in the core must precess in unison. The precession is damped by the mutual friction in the core. This damping has a timescale of tens or hundreds of precession periods, and may be observed over the span of human life. The measurement of the damping timescale would constrain the value of $\delta m_p^*/m_p$ and thus provide information about the strong p-n interactions in the neutron-star core.

While the immediate astrophysical impact of our paper is modest, it presents some novel analytical techniques. In section III, we have developed the theory of slow Alfven waves in a gravitationally stratified uniformly rotating fluid (as is applicable for a neutron star). In section IV, we have analyzed the precession dynamics of a biaxial rigid body in the presence of strong gravitomagnetic field. As far as we are aware, both of these technical developments are new, and we envisage their further applications to the dynamics of neutron stars.

6. ACKNOWLEDGMENTS

We thank Bennett Link, Maxim Lyutikov, Andrew Melatos, and Christopher Thompson for numerous discussions. Our research was supported by NSERC.

REFERENCES

Alpar, A. M., & Sauls, J. A. 1988, ApJ, 327, 723
Alpar, A. M., Langer, S. A., & Sauls, J. A. 1984, ApJ, 282, 533
Baym, G., Pethick, C. J., & Pines, D. 1969, Nature, 224, 673
Blandford, R. D. 1995, JApA, 16, 191
Blandford, R. D., & Thorne, K. S. 2003, *Applications of Classical Physics*, http://www.pma.caltech.edu/Courses/ph136/yr2002/index.html, Chapter 18
Bondi, H., & Gold, T. 1955, MNRAS, 115, 41
Buckley, K. B. W., Metlitski, M. A., & Zhitnitsky, A. R. 2003, astro-ph/0308148
Cordes, J. 1993, *Planets around pulsars*, ASP conference series, vol. 36, pp. 43–60 [Eds: Phillips, Thorsett, Kulkarni].
Cutler, C., Ushomirsky, G., & Link, B. 2003, ApJ, 588, 975
Hide, R., Boggs, D. H., & Dickey, J. O. 2000, GeoJL, 143, 777
Jones, D. I., & Andersson, N. 2002, MNRAS, 324, 811
Jones, P. B. 1998, MNRAS, 296, 217
Kinney, J., & Mendell, G. 2003, Phys. Rev. D, 67, 024032
Levin, Y., & Ushomirsky, G. 2001, MNRAS, 322, 515
Link, B., & Epstein, R. I. 2001, ApJ, 556, 392
Link, B., & Cutler, C. 2002, MNRAS, 336, 211
Link, B. 2002, astro-ph/0211182
Link, B. 2003, Phys. Rev. Letters, 91, 1101
Reisenegger, A., & Goldreich, P., ApJ, 395, 240
Sedrakian, A., Wasserman, I., & Cordes, J. M. 1999, ApJ, 524, 341
Shabanova, T. V., Lyne, A. G., & Urama, J. O. 2001, ApJ, 552, 321
Shaham, J. 1977, ApJ, 214, 251
Stairs, I. H., Lyne, A. G., & Shemar, S. L. 2000, Nature, 406, 484
Thorne, K. S., & Gursel, Y. 1983, MNRAS, 205, 809
Thorne, K. S., Price, R. H., & MacDonald, D. A. 1986, *Black holes: The membrane paradigm*, Yale University Press, New Haven
Wasserman, I. 2003, MNRAS, 341, 1020